TEACHER'S NAME: ______________________

STUDENT'S NAME: ______________________

BAULKHAM HILLS HIGH SCHOOL

YEAR 12

YEAR 12 HALF YEARLY EXAMINATION

2008

MATHEMATICS
EXTENSION 2

Time allowed - Three hours
(Plus five minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Begin each Question on a fresh page.
- All necessary working should be shown. Marks shown are a guide only
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet provided.

Question 1

MARKS

a) If $z = \dfrac{2+i}{2-i}$, find

(i) z in the form a+ib (where a and b are real) **1**

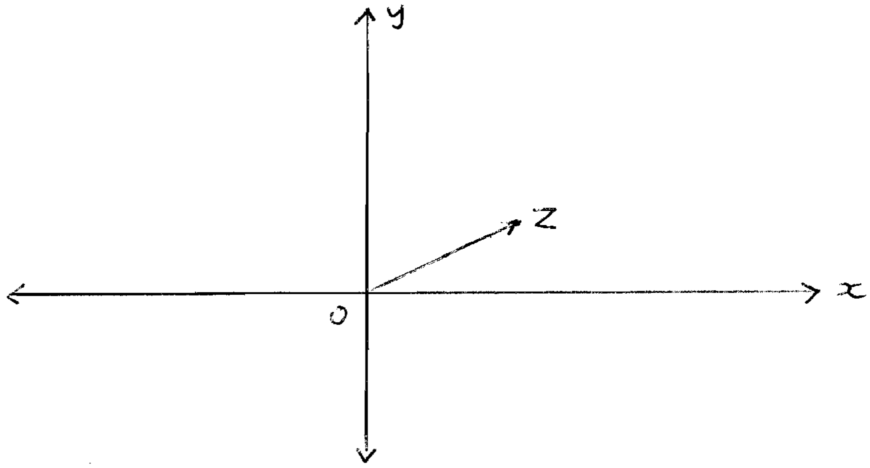
(ii) $|z|$ **1**

(iii) Arg z, correct to the nearest degree **1**

(iv) $\overline{z}$ **1**

(v) Im (iz) **1**

b) The point and vector representing the complex number z are shown on the Argand diagram.



Copy the diagram, and plot and label the points representing $\overline{z}$, $2z$, iz *and* $(2+i)z$ **4**

c) Find the two square roots of 4-3i **3**

d) If $w = \sqrt{3} + i$, express w in modulus-argument form and hence evaluate w^6 **3**

Question 2

a) Sketch on an Argand diagram:

(i) $|z-1| = |z+i|$ **1**

(ii) the region where $|z| \le 1$ *and* $0 \le \arg z \le \frac{\pi}{3}$ **3**

Question 2 (cont)

b) Sketch the locus of z if $|z-(1+i)|=2$. For this locus, what is the maximum value of :

(i) $|z|$

(ii) arg z **3**

c) Use De Moivre's theorem and the expansion of $(\cos\theta + \mathbf{i}\sin\theta)^3$ to obtain an expression for $\cos^3\theta$, and hence find $\int \cos^3\theta\,.d\theta$ **5**

d) $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Prove that $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ **3**

Question 3

a) If $x^2 - y^2 = 0$, show that $\frac{dy}{dx} = \frac{x}{y}$ and express $\frac{d^2y}{dx^2}$ in its simplest form. **3**

b) (i) Sketch the ellipse $\frac{x^2}{16} + y^2 = 1$, showing the intercepts, foci and directrices **4**

(ii) Write down the parametric equations of this ellipse **2**

(iii) Find the equation of the tangent to this ellipse at the point on it where x=2 and y>0. **3**

c) Sketch the locus of the complex number z=x+iy if $|z-3| + |z+3| = 10$.
Find the Cartesian equation of this locus. **3**

Question 4

a) P(ct , $\frac{c}{t}$) and Q ($\frac{ct}{3}$, $\frac{3c}{t}$) are two points on the rectangular hyperbola $xy = c^2$ (t>0).

(i) Sketch the hyperbola and draw and label the chord PQ. **1**

(ii) Find the coordinates of M, the midpoint of PQ. **2**

(iii) Show that the locus of M as P and Q vary is a branch of a hyperbola. **3**

(iv) Show that the the equation of the tangent at P is $x + t^2y = 2ct$. **3**

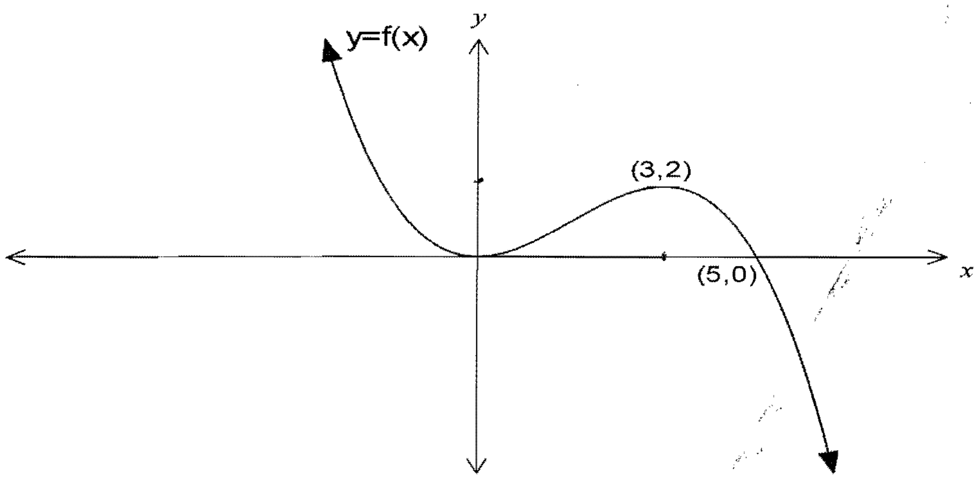
(v) The tangent at P meets the x-axis at A and the y-axis at B. Prove that the area of ΔOAB is constant. **3**

b) Find the value of k if $y = e^{kx}$ is a solution of $y'' - y' - 20y = 0$. **3**

Question 5

a) A relation is defined by $x^2 + xy - 2y^2 = 0$. Show that its gradient has only two values, 1 and $-\frac{1}{2}$. Hence or otherwise, sketch the relation. **3**

b) The diagram shows $y = f(x)$



Draw neat large sketches of

(i) $y = \frac{1}{f(x)}$ **3**

(ii) $y = [f(x)]^2$ **2**

(iii) $y = f(x) + |f(x)|$ **2**

(iv) $y = f(|x|)$ **2**

(v) $y = e^{f(x)}$ **3**

Question 6

a) Draw the graph of $y = |2x-1| + |x-2|$ for the domain $-5 \le x \le 5$.
Hence or otherwise, solve $|2x-1| + |x-2| > 6$. **4**

b) Determine the maximum and minimum values of $\frac{1}{a + b\cos\theta}$ if $a > b > 0$. **3**

Question 6 cont

c) For the function $f(x) = \dfrac{x-2}{x^2-1}$:

(i) Determine the x-values of any stationary points and their nature.

(You do not need to find the y-values). **4**

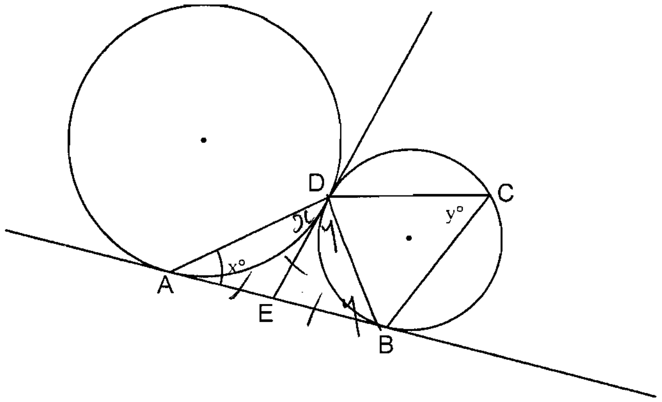
(ii) Sketch $y = f(x)$ showing the important features. **4**

Question 7

a) If $\overline{z_1} + \overline{z_2} = 5 + 2i$, find $z_1 + z_2$ **1**

b) AB and ED are common tangents to the circles shown.

Prove that the angles marked x and y are complementary **5**



c) (i) If $\dfrac{p}{q} = \dfrac{r}{s}$ and $p = rk$, prove that $\dfrac{p+q}{p-q} = \dfrac{r+s}{r-s}$. **2**

(ii) If $\dfrac{p}{q} = \dfrac{\sin\alpha + \cos\alpha}{\sin\alpha - \cos\alpha}$, simplify $\dfrac{p+q}{p-q}$. **2**

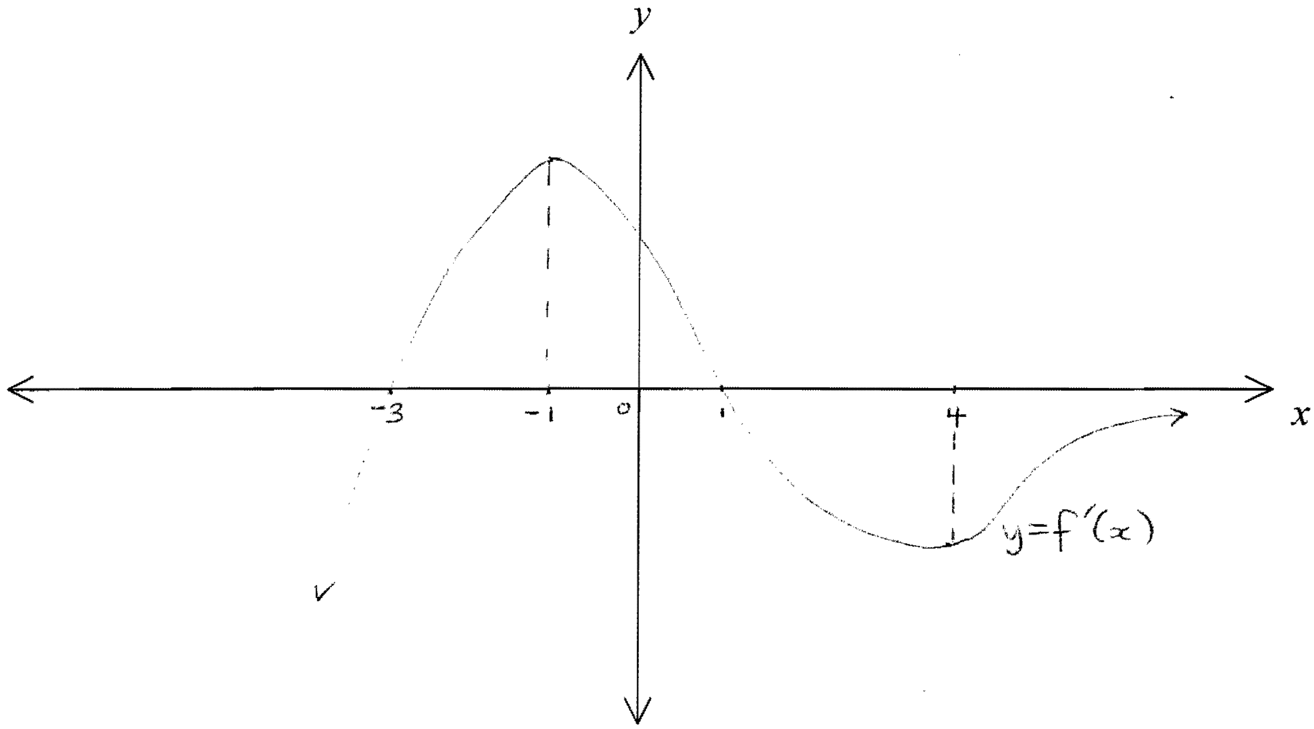
Question 7 (cont)

d) The graph below shows $y = f'(x)$

$f'(-3) = f'(1) = 0.$

As $x \to -\infty$, $f'(x) \to -\infty$.

As $x \to \infty$, $f'(x) \to 0$



Sketch the curve $y = f(x)$ given that f(0)=0, f(-3)=-4, f(1)=2 and f(5)<0. Clearly show the important features. **5**

Question 8

a) PQ is a chord on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangents at P and Q intersect at $T(x_0, y_0)$.

PQ meets a directrix at K.

(i) Draw a diagram showing all the above information **1**

(ii) Write down the equation of the chord of contact from T. *No proof is required.* **1**

(iii) Find the coordinates of K **2**

(iv) S is the focus which is nearer to K. Show that $\angle TSK = 90^\circ$. **3**

b) A and B are angles of a triangle. If sin (A-B) = $\frac{3}{5}$ and sin A cos B = $\frac{4}{5}$,

(i) find the exact value of:

I. cos A sin B **2**

II. sin (A+B) **1**

III. $\frac{\tan A}{\tan B}$ **2**

(ii). Deduce the value of tan B from (II) and (III) above **3**

End of exam

2008 Ext 2 HY Solutions

Q1.

a) (i) $z = \frac{2+i}{2-i} \cdot \frac{2+i}{2+i}$

$= \frac{3+4i}{5}$

$= \frac{3}{5} + \frac{4}{5}i$ (1)

(ii) $|z| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$

$= 1$ (1)

(iii) $\arg z = \tan^{-1}\left(\frac{4/5}{3/5}\right)$

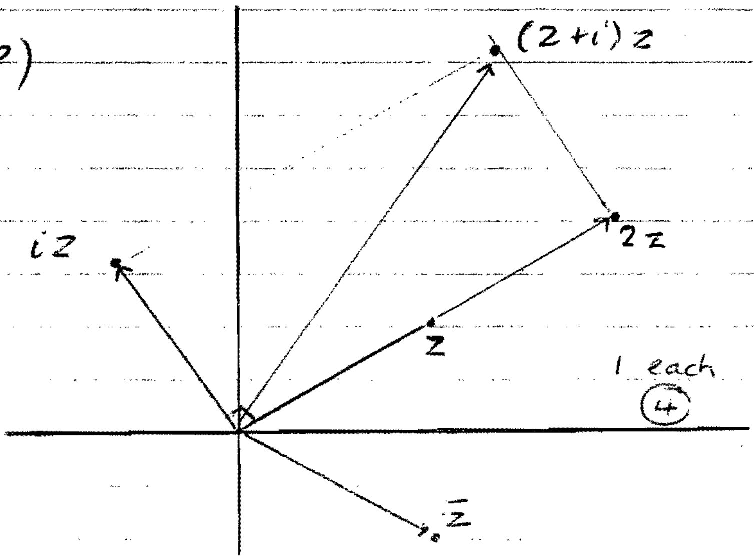
$= 53°8'$ (1)

(iv) $\bar{z} = \frac{3}{5} - \frac{4}{5}i$ (1)

(v) $iz = \frac{3}{5}i - \frac{4}{5}$

$\therefore \text{Im}(iz) = \frac{3}{5}$ (1)

b)



1 each (4)

c) $(x+iy)^2 = 4-3i$

$x^2 - y^2 + 2ixy = 4 - 3i$

$x^2 - y^2 = 4$ ✓

$2xy = -3$

$y = -\frac{3}{2x}$

$x^2 - \frac{9}{4x^2} = 4$

$4x^4 - 16x^2 - 9 = 0$

By quad. formula,

$x^2 = \frac{9}{2}, \cancel{\frac{-1}{2}}$

$x = \pm \frac{3}{\sqrt{2}}$

$y = \mp \frac{1}{\sqrt{2}}$

$\therefore$ Roots are $z = \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

and $z = -\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

d) $w = \sqrt{3} + i$

$= 2 \text{ cis } \frac{\pi}{6}$ mod. (1) arg (1)

$w^6 = 2^6 \text{ cis}\left(6 \times \frac{\pi}{6}\right)$

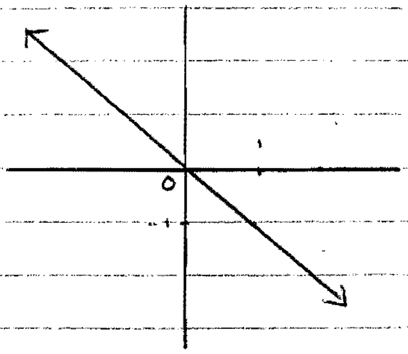
$= 64 \text{ cis } \pi$

$= 64(-1 + 0i)$

$= -64.$ (1)

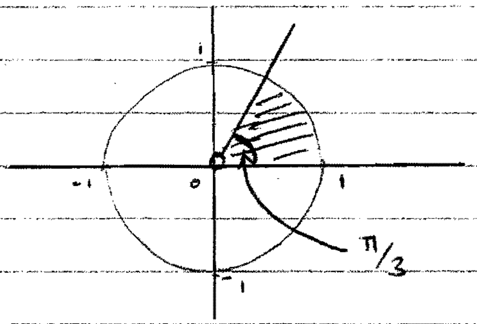
Q2

a) (i)



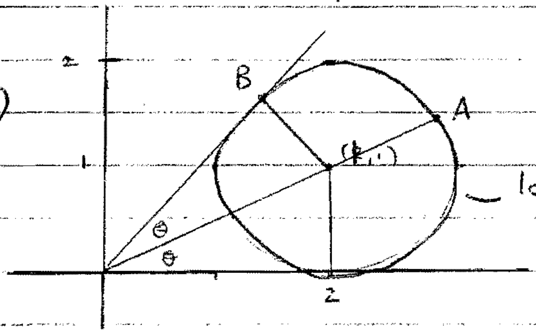
(1)

(ii)



(1) circle
(1) ray
(1) shading + empty circle at 0.

b)



locus (circle) ← (1)

Max $|z| = \sqrt{5} + 1$ (at A) (1)

Max $\arg z = 2\theta$ (at B)

$= 2\tan^{-1}\frac{1}{2}$

$\doteqdot 53°8'$ (1)

[illegible]

c) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta \cdot i\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ (1) (1)

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ (equating reals)

$\underline{\cos^3\theta = \cos 3\theta + 3\cos\theta\sin^2\theta}$ (1)

$\therefore \int \cos^3\theta \, d\theta = \int \cos 3\theta + 3\cos\theta\sin^2\theta \, d\theta$

$= \underline{\frac{1}{3}\sin 3\theta + \sin^3\theta + c}$ (1) (1)

or $\frac{1}{4}\left[\frac{1}{3}\sin 3\theta + 3\sin\theta\right]$

d) $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 (\cos\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2$
$+ i^2 \sin\theta_1 \sin\theta_2)$ ①

$= r_1 r_2 \left((\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2) \right)$ ①

$= r_1 r_2 \cdot \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$ ①

$\therefore \arg z_1 z_2 = \theta_1 + \theta_2$ ①

$= \arg z_1 + \arg z_2$

Q3.

a) $x^2 - y^2 = 0$

$2x - 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$

$\frac{d^2y}{dx^2} = \frac{y \cdot 1 - x \cdot {}^{dy}/_{dx}}{y^2}$

$= \frac{y - x \cdot \frac{x}{y}}{y^2}$

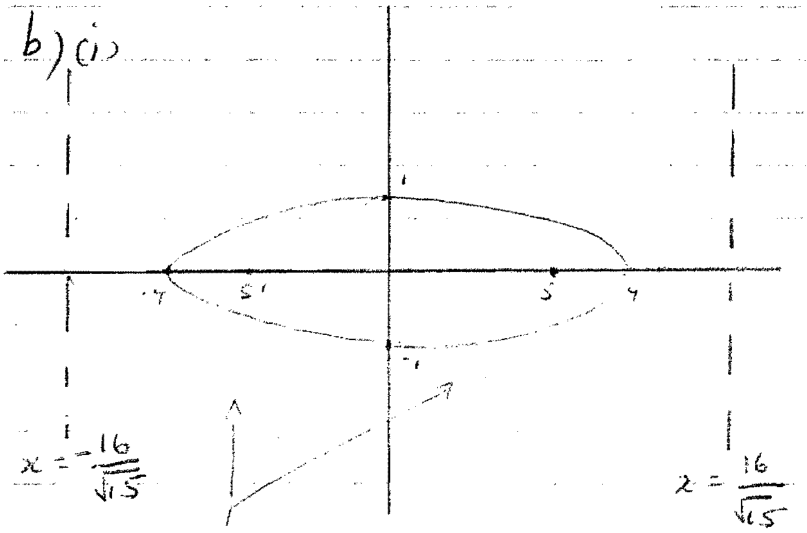
$= \frac{y - \frac{x^2}{y}}{y^2}$

$= \frac{y^2 - x^2}{y^3}$ but $x^2 - y^2 = 0$, $y^2 - x^2 = 0$

$= \frac{0}{y^3}$

$= 0$

b) (i)



$S(\sqrt{15}, 0)$ x-int (1)

$S'(-\sqrt{15}, 0)$ y-int (1)

Foci (1)

Dir (1)

(ii) $x = 4\cos\theta$ (1)

$y = \sin\theta$ (1)

(iii) When $x = 2$, $\frac{4}{16} + y^2 = 1$

$y^2 = \frac{12}{16}$

$y = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$ $(y > 0)$

So pt of contact is $\left(2, \frac{\sqrt{3}}{2}\right)$ (1)

$\frac{2x}{16} + 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x/8}{2y}$

$= -\frac{x}{16y}$

$= -\frac{2}{16 \cdot \frac{\sqrt{3}}{2}}$

$= -\frac{2}{8\sqrt{3}}$

$= \frac{-1}{4\sqrt{3}}$ (1)

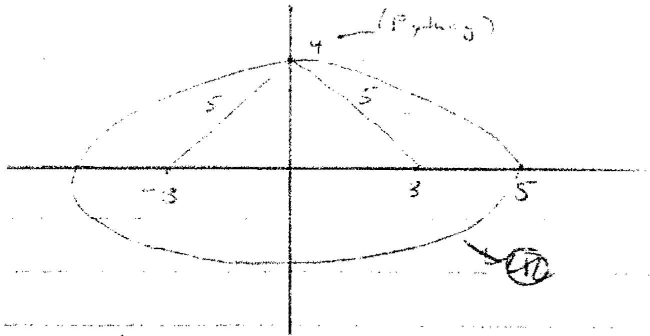
$y - \frac{\sqrt{3}}{2} = \frac{-1}{4\sqrt{3}}(x - 2)$

$4\sqrt{3}\left(y - \frac{\sqrt{3}}{2}\right) = -x + 2$

$4\sqrt{3}y - 6 = -x + 2$

$x + 4\sqrt{3}y - 8 = 0$. (1)

c)

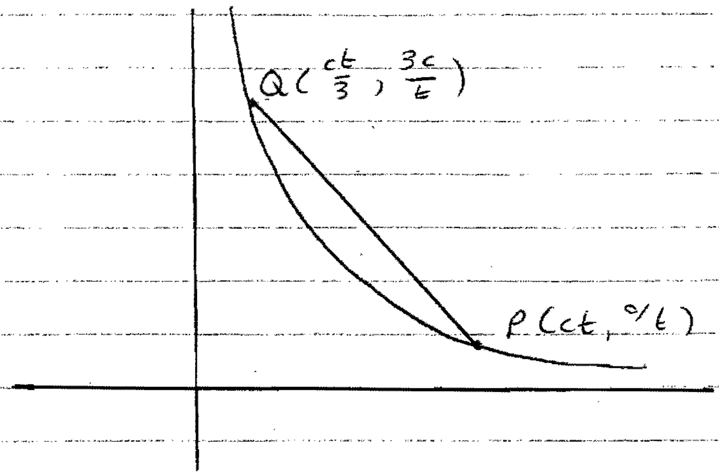


$\frac{x^2}{25} + \frac{y^2}{16} = 1$ An ellipse (1)

Eqn (2)

Q4.

a) (i)



Note: Q is to left of P.

1

(ii) $M = \left(\dfrac{ct + \frac{ct}{3}}{2}, \dfrac{\frac{3c}{t} + \frac{c}{t}}{2} \right)$

$= \left(\dfrac{\frac{4ct}{3}}{2}, \dfrac{\frac{4c}{t}}{2} \right)$

$= \left(\dfrac{2ct}{3}, \dfrac{2c}{t} \right)$ 1 x-value, 1 y-value

(iii) Since $x = \dfrac{2ct}{3}$, $y = \dfrac{2c}{t}$

$xy = \dfrac{2ct}{3} \cdot \dfrac{2c}{t} = \dfrac{4c^2}{3}$ 1 which is constant.

Since xy is constant 1 locus is a hyperbola. Since $t > 0$, only ~~the only~~ one branch. 1

(iv) Tangent $\dfrac{dy}{dx} = -c^2 x^{-2}$ $\left(y = \dfrac{c^2}{x} \right)$

$= -\dfrac{c^2}{x^2}$ 1

$= \dfrac{-c^2}{(ct)^2}$ at $P(ct, c/t)$

$= \dfrac{-c^2}{c^2 t^2}$

$= -\dfrac{1}{t^2}$ 1

$y - c/t = -\dfrac{1}{t^2}(x - ct)$

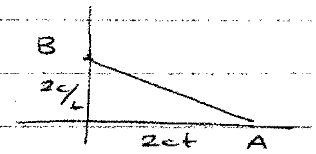
$t^2(y - c/t) = -x + ct$ 1

(v) Tang. at P $\quad x + t^2 y = 2ct$

At A : $\quad y = 0 \quad x = 2ct \qquad A(2ct, 0)$ 1

At B : $\quad x = 0 \quad t^2 y = 2ct$

$$y = \frac{2c}{t} \qquad B\left(0, \frac{2c}{t}\right) \quad 1$$



$$\text{Area} = \frac{1}{2} bh = \frac{1}{\cancel{2}} \cdot \cancel{2}c\cancel{t} \cdot \frac{2c}{\cancel{t}}$$

$= 2c^2$ which is constant. 1

b) $y = e^{kx}$

$$\left.\begin{aligned} y' &= ke^{kx} \\ y'' &= k^2 e^{kx} \end{aligned}\right\} 1$$

$\therefore \; y'' - y' - 20y = (k^2 - k - 1)e^{kx} = 0$ 1 only when

$$k^2 - k - 1 = 0$$

$$k = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}. \quad 1$$

Question 5.

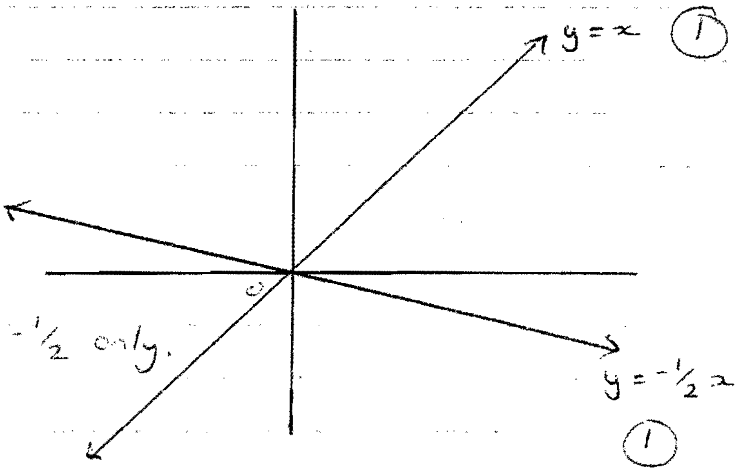
a) $(x + 2y)(x - y) = 0$

$x = -2y \qquad x = y$

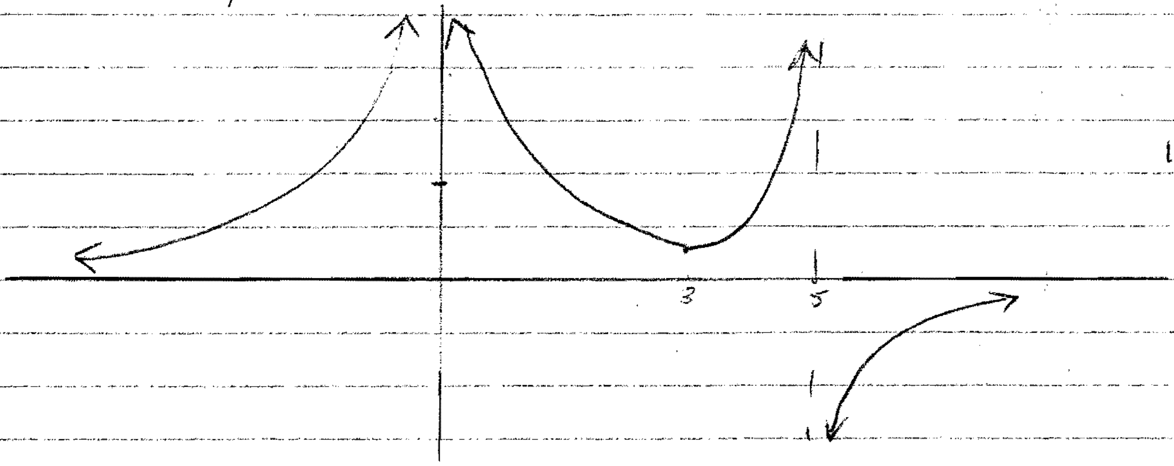
$y = -\frac{1}{2}x \qquad \underline{(m = 1)}$

$\underline{(m = -1/2)} \qquad 1$

So $m = \frac{dy}{dx} = 1$ or $-1/2$ only.



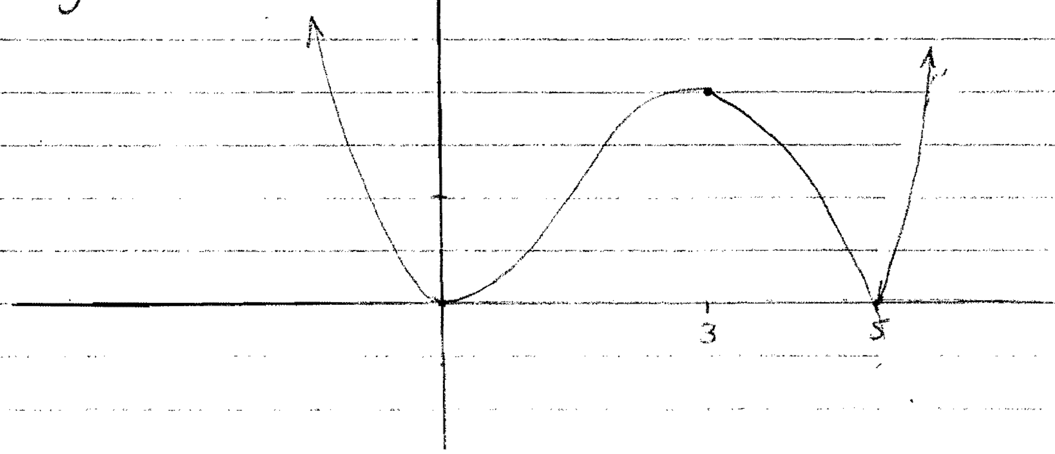
(i) $y = \frac{1}{f(x)}$



3

1 each section

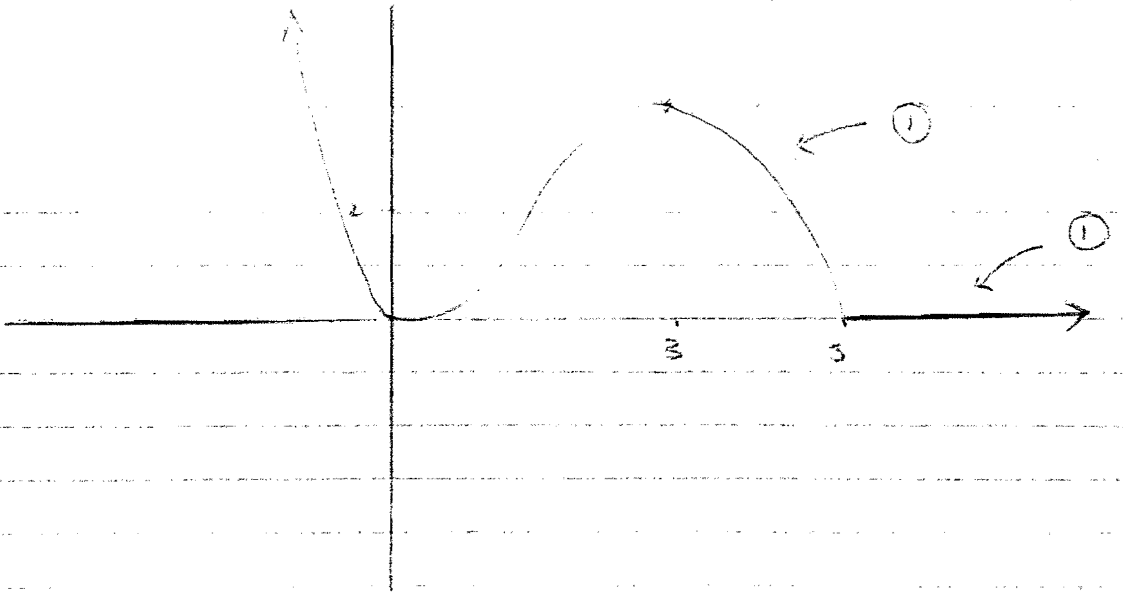
(ii) $y = [f(x)]^2$



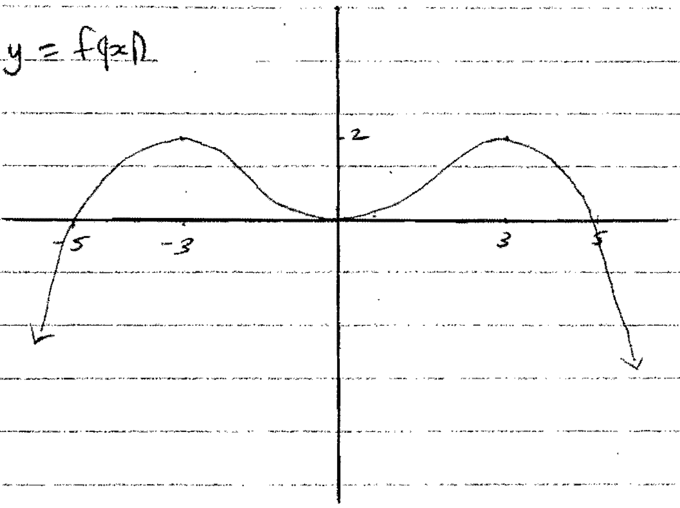
2

(iii) $y = f(x) + |f(x)|$

$= 0$ when $f(x)$ neg
$= 2f(x)$ when $f(x)$ pos

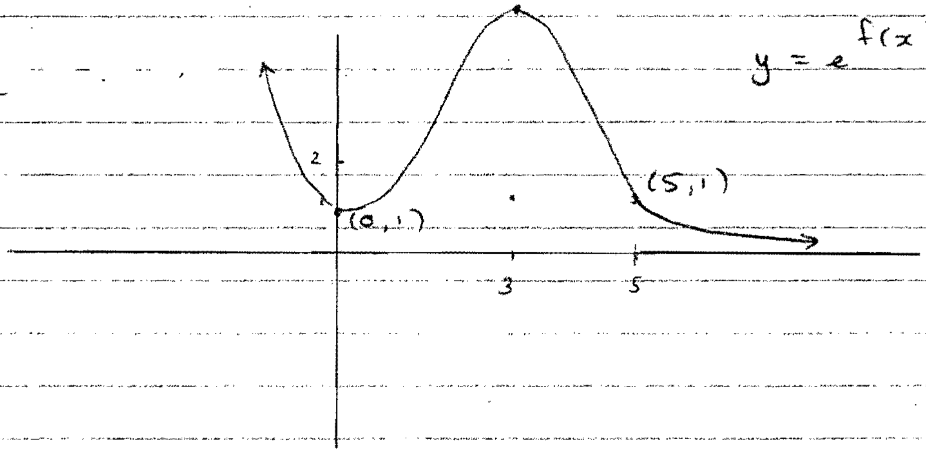


(iv) $y = f(|x|)$



(2)

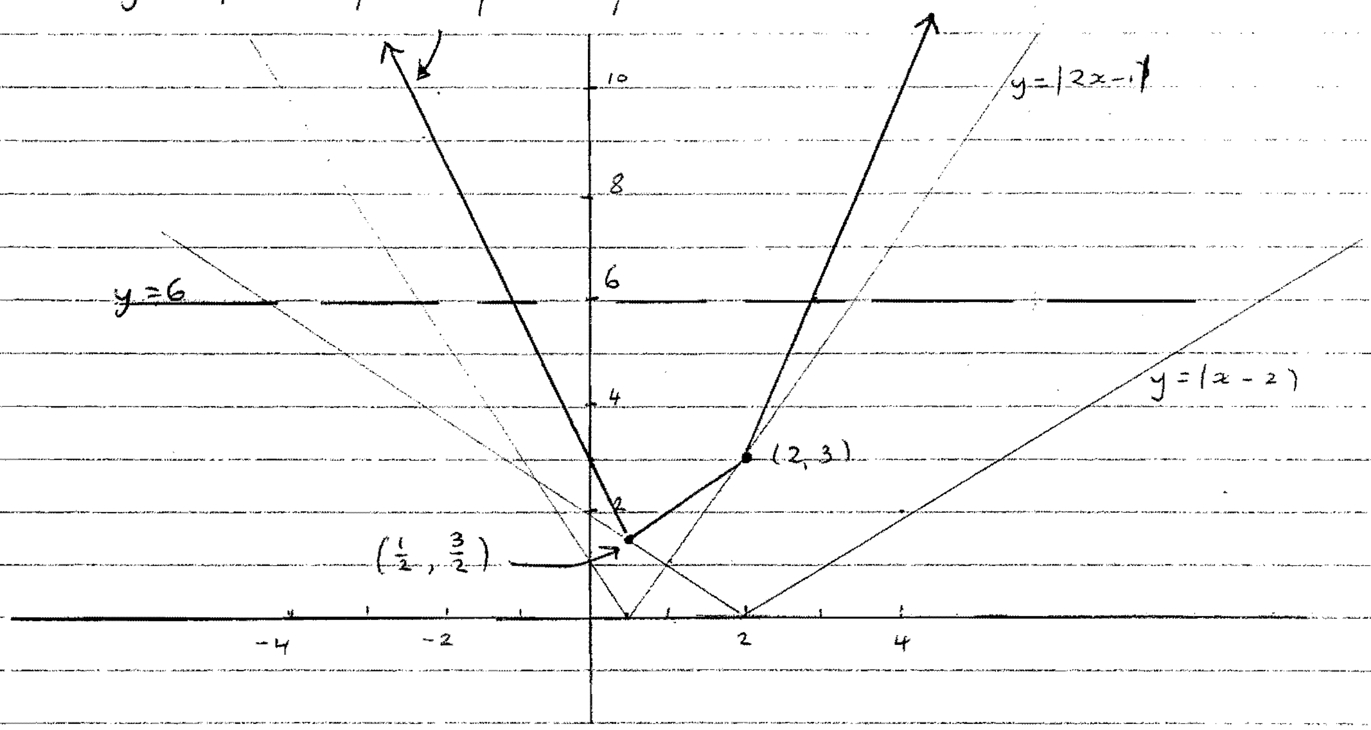
(v) $y = e^{f(x)}$



(3)

Q6.

a) $y = |2x-1| + |x-2|$



y=6 Intersects when $x=-1$, $x=3$ (Can solve eqns or check by substitution)

Soln to inequality : $\underline{x<-1}$, $\underline{x>3}$.

b) For $a+b\cos\theta$, max value $= a+b$, min value $= a-b$ } and both results are positive

∴ For $\frac{1}{a+b\cos\theta}$: max value $= \frac{1}{a-b}$

min value $= \frac{1}{a+b}$

c) $f(x) = \frac{x-2}{x^2-1} = \frac{x-2}{(x-1)(x+1)}$

(i) $f'(x) = \frac{(x^2-1).1-(x-2).2x}{(x^2-1)^2} = \frac{x^2-1-2x^2+4x}{(x^2-1)^2}$

$= \frac{-x^2+4x-1}{(x^2-1)^2}$

For stat pts $f'(x) = 0$: when $x^2-4x+1=0$

$x = \frac{4 \pm \sqrt{16-4}}{2}$

$= \frac{4 \pm 2\sqrt{3}}{2}$

$= 2 \pm \sqrt{3}$

Use the first deriv. to determine nature

When $x = 2+\sqrt{3} \doteq 3.7$

x	3	$2+\sqrt{3}$	4
$f'(x)$	$+$	0	$-$
	$-9+12-1$		$-16+16-1$

[Sign of $f'(x)$ = Sign of $-x^2+4x-1$]

Max turn pt when $x = 2+\sqrt{3}$

When $x = 2-\sqrt{3} \doteq 0.3$

x	0	$2-\sqrt{3}$	1
$f'(x)$	$-$	0	$+$
	$-0+0-1$		$-1+4-1$

Min turn pt when $x = 2-\sqrt{3}$

(ii) $f(x) = \dfrac{x-2}{x^2-1} = \dfrac{x-2}{(x-1)(x+1)}$

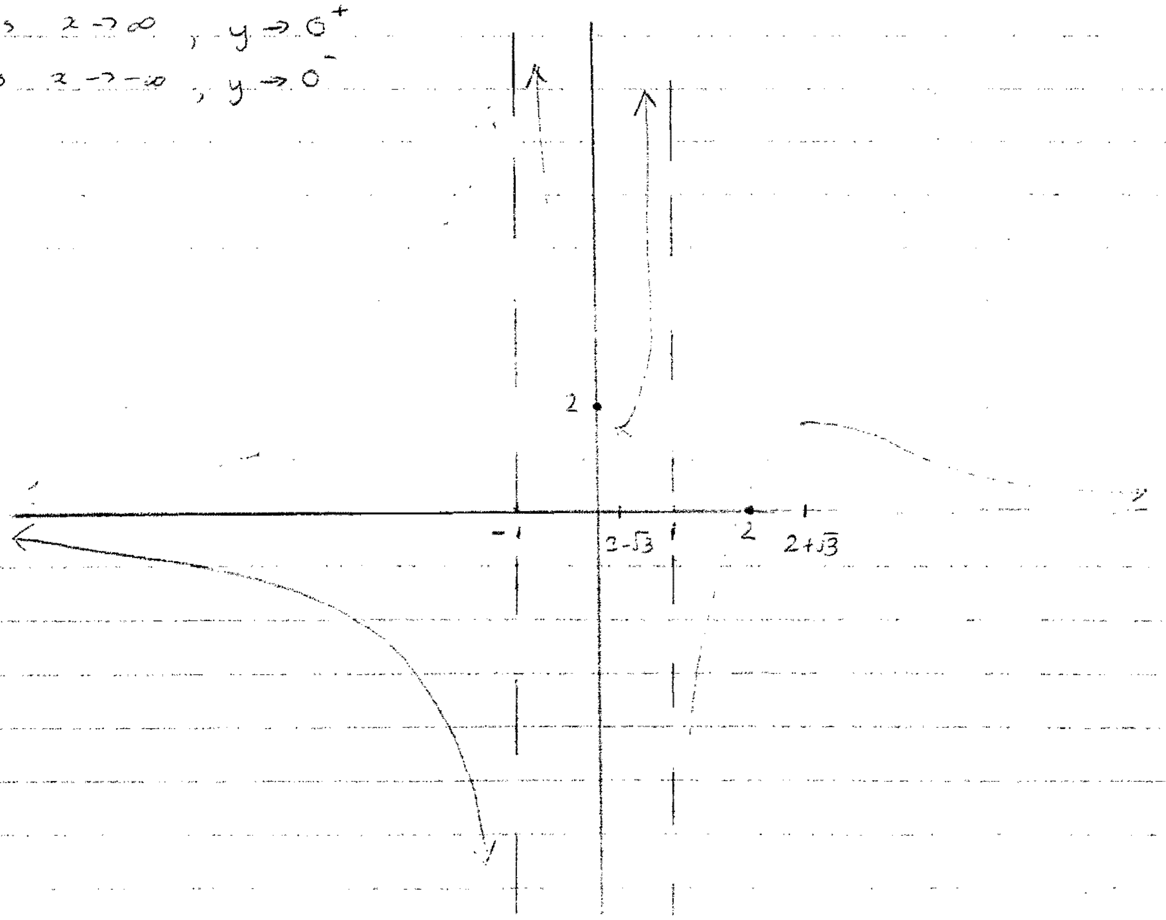
Vert asymp. $x = 1, -1$

Horiz. asymp. $y = 0$

Stat pts as above

As $x \to \infty$, $y \to 0^+$

As $x \to -\infty$, $y \to 0^-$



Q7.

a) $5 - 2i$

b) Join DB. Draw circle with centre E, through A, D, B. (1)

$\angle ADE = x$ (base $\angle$s, isos Δ) (1)

$\angle DEB = 2x$ (ext $\angle$ of Δ = sum of int. opp $\angle$s) (1)

$\angle EBD = \angle EDB$ (base $\angle$s, isos Δ)

$= \dfrac{180 - 2x}{2}$ ($\angle$ sum of Δ)

$= 90 - x$ (1)

$y = 90 - x$ ($\angle$ between tangent + chord = $\angle$ in alt. seg.) (1)

$\therefore x + y = 90$

c) (i) $\dfrac{rk}{q} = \dfrac{r}{s}$ (since $\underline{p = rk}$)

$\dfrac{k}{q} = \dfrac{1}{s}$

$\therefore \underline{q = sk}$

$$\frac{p+q}{p-q} = \frac{rk + sk}{rk - sk} = \frac{k(r+s)}{k(r-s)}$$

$$= \frac{r+s}{r-s}$$

(ii) If $\dfrac{p}{q} = \dfrac{\sin\alpha + \cos\alpha}{\sin\alpha - \cos\alpha}$ $\leftarrow kr$ $\leftarrow ks$

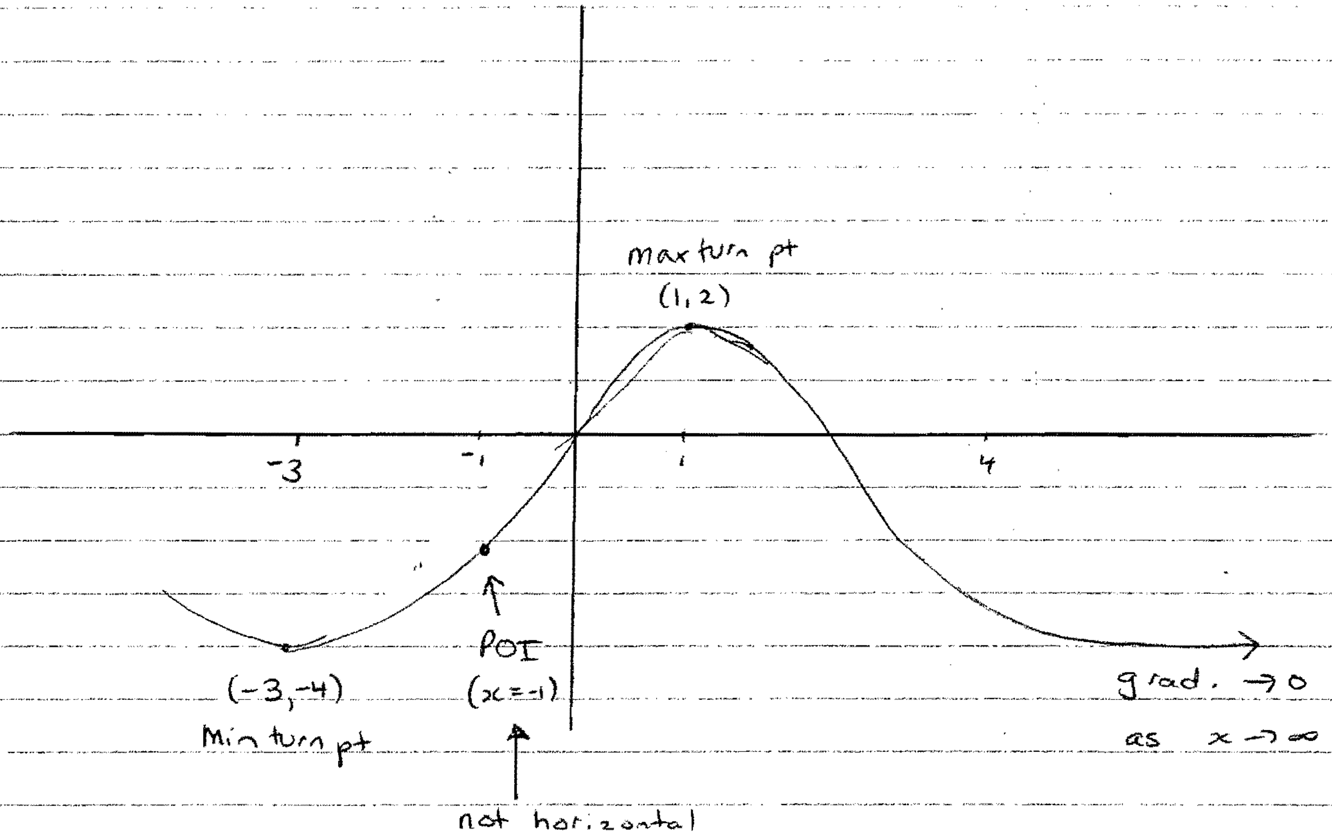
then $\dfrac{p+q}{p-q} = \dfrac{kr + ks}{kr - ks}$

$$= \frac{\sin\alpha + \cos\alpha + \sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha - \sin\alpha + \cos\alpha}$$

$$= \frac{2\sin\alpha}{2\cos\alpha}$$

$$= \tan\alpha$$

d)



Thr (0, 0) ①

Min (-3, -4) ①

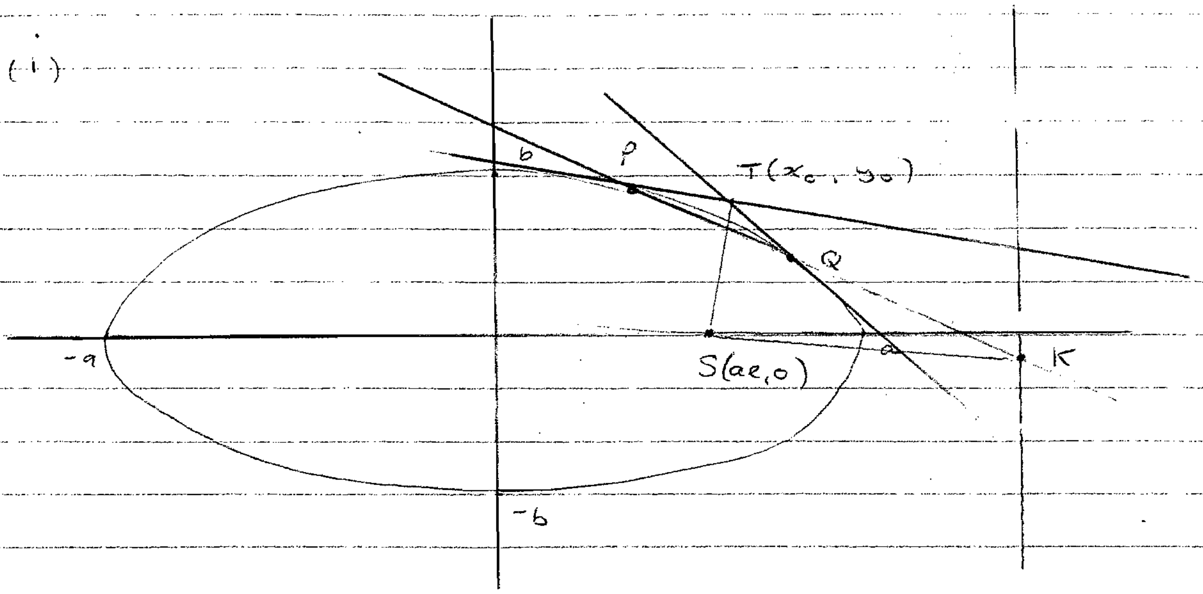
Max (1, 2) ①

POI, not horiz ①

y neg + m → 0 as x → ∞ ①

Q8.

a) (i)



(ii) $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$

(iii) At K : $x = a/e$

$$\frac{\frac{a}{e} \cdot x_0}{a^2} + \frac{yy_0}{b^2} = 1$$

$$\frac{x_0}{ae} + \frac{yy_0}{b^2} = 1$$

$$y = \left(1 - \frac{x_0}{ae}\right) \cdot \frac{b^2}{y_0}$$

$$\therefore K\left(\frac{a}{e}, \frac{b^2}{y_0}\left(1 - \frac{x_0}{ae}\right)\right)$$

(iv) $m_{TS} = \dfrac{y_0 - 0}{x_0 - ae} = \dfrac{y_0}{x_0 - ae}$

$$m_{SK} = \frac{\frac{b^2}{y_0}\left(1 - \frac{x_0}{ae}\right) - 0}{\frac{a}{e} - ae}$$

$$= \frac{\frac{b^2}{y_0} \cdot \frac{ae - x_0}{ae}}{\frac{a - ae^2}{e}} \times \frac{e}{e}$$

$$= \frac{\frac{b^2(ae - x_0)}{ay_0}}{a(1 - e^2)}$$

but $b^2 = a^2(1 - e^2)$

$$= \frac{\cancel{b^2}(ae - x_0)}{\cancel{a^2} y_0 (\cancel{1 - e^2})}$$

$$= \frac{ae - x_0}{y_0}$$

$$m_{TS} \times m_{SK} = \frac{\cancel{y_0}}{\cancel{x_0 - ae}} \times \frac{\cancel{ae - x_0}^{-1}}{\cancel{y_0}} = -1$$

$\therefore \angle TSK = 90°$

b) (i) (I) $\sin(A-B) = \underbrace{\sin A \cos B}_{\frac{4}{5}\text{ given}} - \underbrace{\cos A \sin B}_{\frac{1}{5}\text{ follows.}} = \frac{3}{5}$ given   expansion

~~(I)~~ $\cos A \sin B = 1/5$

(II) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \frac{4}{5} + \frac{1}{5}$$

$$= 1$$

(III) $\frac{\tan A}{\tan B} = \frac{\sin A}{\cos A} \div \frac{\sin B}{\cos B}$

$$= \frac{\sin A \cos B}{\cos A \sin B}$$

$$= \frac{4/5}{1/5}$$

$$= 4.$$

(ii) $\frac{\tan A}{\tan B} = 4$

$\tan B = \frac{1}{4} \tan A$

$\sin(A+B) = 1$ and A, B are $\angle$s of a Δ

$A + B = 90°$ ~~, as 45°~~ only

$A = 90° - B$

$\tan B = \frac{1}{4} \tan(90 - B)$

$\tan B = \frac{1}{4} \cot B$ using complementary ratios

$\tan B = \frac{1}{4 \tan B}$ using reciprocal ratios

$\tan^2 B = \frac{1}{4}$

$\tan B = \pm\frac{1}{2}$ but since $A+B = 90°$ and A, B are $\angle$s of $\triangle$

$B = 30°$, ~~$150°$~~

$\tan B = \frac{1}{2}$ only.